Algebra as language: Wallis and Condillac on the nature of algebra

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El presente artículo estudia un episodio concreto en el proceso que transformó las notaciones algebraicas de los siglos XVI y XVII en un poderoso *lenguaje formal*. En el mismo se comparan las ideas de Wallis y de Condillac sobre la naturaleza del álgebra con el objetivo de entender las diferencias entre el todopoderoso lenguaje algebraico de las primeras décadas del siglo XVIII y las limitaciones inherentes a las notaciones algebraicas del siglo XVII. De forma particular, se valora aquí el papel desempeñado por el elemento gráfico de las notaciones algebraicas, así como su simetría en algunos casos particulares, en la manera como Wallis entendía el poder demostrativo de las notaciones algebraicas. El artículo arguye que la debilidad del álgebra *qua* lenguaje formal en Wallis es consistente con la importancia que él le acordaba como instrumento de descubrimiento. También es consistente con la opinión de Wallis que el álgebra no altera de forma substancial las demostraciones clásicas. Finalmente, y quizá de forma más importante, esta debilidad también es consistente con la idea que el álgebra sólo era una taquigrafía que "condensaba" los argumentos matemáticos y "los mostraba en una sola mirada".

The present article focuses on a specific stage in the process through which algebraic notations became a powerful formal *language*. The article compares the views of Condillac and Wallis on the nature of algebra as a way to understand the differences between the all powerful algebraic language of the first decades of the 18th century and the limitations of 17th-century algebraic notations. In particular, it considers the role the graphical element of algebraic notations, and their symmetry in some specific cases, played in John Wallis's understanding of the demonstrative power of the algebraic notations. It shows that the weakness of Wallis's algebra *qua* formal language is fully consistent with his emphasis on algebra as a tool of discovery. It is also consistent with Wallis's view that algebra does not produce "new" demonstrations, or that algebra does not alter the substance of the classical demonstrations. Finally, and perhaps more significantly, such a weakness is also consistent with the notion that algebra was but a tachigraphy that "condensed" mathematical arguments and "offered them at one glance".

Introduction

The present article focuses on a specific stage in the process through which algebraic notations became a powerful formal language. In particular, it considers the role the graphical element of algebraic notations, and their symmetry in some specific cases, played in Wallis's understanding of the demonstrative power of the algebraic language. As is well

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known, John Wallis (1616-1703), one of the staunchest defenders of the then novel "algebraic style", was one of the mathematicians most innovative in using algebra. As we shall see, Wallis did not have at his disposal a self-sustaining algebraic language, that is to say, one that could fully sustain formal or symbolic deductions. On the other hand, Wallis's algebraic notations had strength (even from a deductive point of view) derived from considerations of symmetry and graphical suggestiveness.

Two questions must be previously discussed in order to clarify the context of Wallis's contributions. One concerns the usual view that sees algebra and algebraic notations as antagonistic to geometry during the 17th century. The second one concerns the very notion of algebraic language as it took shape and consistency in the first decades of the 18th century. We shall start by addressing these two preliminary points and then will concentrate in Wallis's understanding of algebra and his use of the algebraic "style".

The complicated relationship between algebra and geometry sustained throughout the seventeenth century is usually portrayed in terms of conflict, struggle, and opposition. As Morris Kline famously said, in the 17th century "algebra did rise above the limitations imposed by geometric thinking". Until quite recently most historians of mathematics saw 17th-century geometry as a constraint, an obstacle, and a residue of the past. Interestingly, the language of "freedom" is quite common to oppose algebraic to geometric styles and arguments. Thus, according to Kline, Wallis played a crucial role "in freeing arithmetic and algebra from geometric representation" (Kline, 1972 : 279, 281). In the last few years, having realized the complexity of the many ways in which algebra was linked to geometry during the 17th century, historians of mathematics have revised the foregoing view. Yet, somehow the basic idea of the confrontation between rivals remains. Somehow, it is still being suggested that algebra and geometry could hardly coexist—that it was all one or the other. To quote from a competent, very recent book on the history of 17th-century algebra:

Hobbes and Barrow were on the losing side of one of the most important mathematical battles of their century—that pitting the new algebraists against mathematicians linked ... to the geometric tradition of Western mathematics. (Pycior, 1997 : 166).

Although this old cliché contains some truth (as all clichés do), it must be thoroughly revised. In my view, rather than two conflicting views about the purpose and the actual working of 17th-century mathematics, geometry and algebraic analysis were rather two layers of the mathematical discourse. As we shall see today, the rise of symbolic algebra in the second half of the 17th century provides more evidence of the foundational role geometry that was still playing then in the work of the most advanced algebraists.

The "language" of algebra in the early Enlightenment

With a few exceptions on minor points, the formal system of algebraic symbols stabilizes and practically acquires its final modern form (even in terms of graphical solutions and conventions) by the end of the first third of the 18th century. Even more important, it was by then generally and explicitly recognized as a *language*. By then, "analysis" (in the particular sense so well described by Marco Panza, Craig Fraser, and others) was almost coextensive with mathematics, and the language of algebraic symbols was the only one (almost) all mathematicians used.[1] By then, it had become commonplace that "algebraic" or "analytic" demonstrations (meaning demonstrations involving but the manipulation of algebraic symbols according to formal rules, like for instance demonstrating that $(a \pm b)^2 = a^2 + b^2 \pm 2ab$ by the formal product of $a \pm b$ by itself) were more direct, more evident, more general, more rigorous, and in all senses to be preferred to a geometrical proof. Proclaimed in many places and ways, the view that algebra was more general, abstract, and fundamental than geometry was given pride of place in D'Alembert's authoritative "Preface" to the *Encyclopédie* (1750) (D'Alembert, 1750 : v-viii). That such a programmatic and carefully constructed text as D'Alembert's "Preface" dealt with this particular issue in the philosophy of mathematics should warn us about its novelty and about how important it was for the philosophy of science of the Enlightenment.

In fact, the *fundamental* role accorded in the 18th century to the algebraic language as well as the 18th-century identification of mathematics with "analysis" both came hand in hand with a new philosophy of knowledge *and* a new philosophy of language. Highly characteristic of the science of the Enlightenment, such novel views on knowledge and language found their more popular and cogent expression in the works of Etienne Bonnot de Condillac (1715-1780). As is well known, Condillac provided systematic expression to Locke's views while complementing them with his own insights. He was the best and most influential expositor of empirical sensationalism, and something like the official philosopher of the French *philosophes*.[2] Condillac argued that whatever we know (either about the material world, about human nature, or about any other

[1] See Fraser (1989), Fraser (1997), Panza (1997).

[2] His most influential works appeared around the middle of the century: *Essai sur l'origine des connaissances humaines* (1746), *Traité des systèmes* (1749), and *Traité des sensations* (1754). His ideas about language and thought (first published in the *Essai sur l'origine des connaissances humaines*) appeared in condensed form in *La logique* (1780). He developed them in his *La langue des calculs* (Condillac, 1981), which he left unfinished and was published only posthumously. Our quotations come from *La logique* on account of both, its having a more finished form, and its representing Condillac's first and earlier views of the first decades of the 18th century. There is no good account of Condillac's life and work, but see G. Le Roy's "Introduction" to his edition of *Oeuvres philosophiques de Condillac*, 3 vol. (Paris: P.U.F., 1947-1951); Gillispie, 1960) : 163-9; Hallie, 1967; Knobloch, 1980; Aarsleff, 1982a.

domain) is known through ideas in our minds that ultimately originate in the sensations perceived by our bodily senses. Our knowledge is but the result of *analyzing* our ideas, which meant finding how they are composed of more simple ideas —he provided the example that we fully know or "understand" a machine when we can disassemble it in its constituent gears, rods, screws, etc. so as to be able to recomposing with them the original machine. Similarly, understanding a difficult, complex idea meant isolating and identifying the simple ideas compounding it. Some ideas, like the one we identify and represent by the word "force", cannot be clearly represented to the imagination. Nonetheless, we can effectively and rigorously use it once we analyze it and know exactly the rapports it keeps with other ideas (like mass, acceleration, etc). It is not necessary to know the real material *essence* of our ideas (we may never know exactly *what* "forces" are in themselves, for example), since all we need is to know how forces work, that is to say their exact relationships to other ideas (Condillac, 1780 : 378-83). In other words, the relationship of our ideas to the "real" things in the material world that correspond to them is bound to be problematic.

Since we may find impossible to discover the real nature of many things (of which me nevertheless have clear ideas), the essential tool for handling and analyzing our ideas is the *words* that identify them in our minds and the rules we have to combine and relate words. Hence, the crucial role the philosophy of science of the Enlightenment accords to language, and the particularly crucial role algebraic language assumes within mathematics during the Enlightenment.[3] Condillac claims that we only think through words, and that all philosophical errors come from the misuse of words —most usually from the use of words whose meaning has not been properly clarified (Condillac, 1780 : 394). Reasoning started with language, and the art of reasoning (which is in essence equivalent to the art of analyzing) progresses just in the measure in which languages do. Without a good, improved language, we can neither analyze our ideas nor analyze the sensory inputs from which our ideas arise. Words are signs for our ideas. We deal with ideas through words, which is the only way we can handle them (*ibid* : 396, 397).

Condillac criticizes previous philosophies of language for not recognizing the crucial role words play in our thinking. Until then, according to him, words had been considered mere instruments for the *communication* of thoughts, but not part and parcel of the activity of thinking. That is, thinking had been supposed to be previous to, independent from, and above its linguistic expression. According to Condillac, however, the art of well reasoning is nothing but the art of speaking correctly (*ibid* : 399, 400, 401). Hence, the foremost importance of a well

[3] See Gillispie (1960), p. 169-72, 233-50; Rider (1990), Beretta (1993), p. 27-59.

designed and well articulated language for philosophical investigations —and the secondary role imagination plays in them (*ibid* : 402). Condillac also criticizes the prominent role previously accorded to definitions and to the synthetic or deductive organization of knowledge, which he attributed to the influence of classical geometry. Not even for teaching purposes is the axiomatico-deductive organization to be recommended, according to him. Its order is only apparent, since it is an artificial one invented by philosophers, not one arising from the actual investigation and discovery of truths; it tires the mind without in any way illuminating it (*ibid* : 403-5).

Condillac is nowadays credited with having substantially contributed to modern theories of language by stressing the intimate connection between language and thought (Aarsleff, 1982b). He was essentially correct in pointing out that Aristotelian, medieval, and 17th-century rationalist theories of language all agreed in taking language to be the expression of already existing logical structures. According to such views (present, for instance, in Arnauld's influential *Logique*), words were signs somehow fitted to already developed ideas, and language expressed already formulated judgments. Words were unavoidable for communication (and thereby they were attached to ideas), but if anything, words were the source of confusion and error in our thoughts.[4] In accordance with then prevalent theories of language, in 1634 the mathematician Pierre Herigone (*fl.* 1630-1640) advocated a radical reform of the mathematical style that incorporated a huge measure of new algebraic notations. He claimed that his aim was to liberate mathematics from the ambiguities and confusion of *all* languages whatever: "i'ay inventé une nouvelle methode de faire les demonstrations, briefve et intelligible, sans l'usage d'aucune langue".[5] We will return below to this opposition of algebra and language typical of the seventeenth century.

Condillac ended his *Logique* by crowning the algebraic language king of the philosophical languages and recommending that as far as possible all sciences should adopt it (*ibid* : 407-8). Algebraic expressions show transparently all the links between the different magnitudes relevant to the problem. Algebraic proofs move from step to step in a long chain of connected expressions in which we find all the clarity and evidence that philosophical arguments might ever achieve. Algebra, claims Condillac, is not (as some mathematicians say) "similar" to, or "sort of" a language, but is in itself a real language that translates reasonings that use words into chains of formal expressions in which letters and others symbols occupy the place of words. Algebra is both an analytic method and a lan-

[4] See, for instance, Arnauld, Nicole (1662), p. 40-1, 42-3, 83, 103-4.

[5] Herigone (1634-42), I, unnumbered preface "Ad lectorem / Au lecteur", p. [2]. On Herigone, see Folkerts, Knobloch, Reich (eds.) (2001), pp. 13f.

guage. All languages are analytic methods if properly constituted, but algebra, being the more precise and well articulated of all existing languages, is the best of all available analytic methods (*ibid*).

Such an understanding of the status and apodictic power of the formal language of algebra was indeed new. In the last decades of the seventeenth century, the foundations of algebraic operations were either taken for granted and not properly discussed (that was essentially Wallis's position) or else they were explicitly grounded on geometric objects and results. The crucial role of the geometric referent for algebraic operations is most evident in the *Essai de Logique* that the natural philosopher Edme Mariotte (*c.* 1620-1684) wrote in collaboration with Roberval (published in 1678). Here Mariotte claims that since algebraic operations are grounded on propositions from arithmetic and geometry, algebra cannot prove basic propositions of arithmetic or geometry —lest we fall in a circular situation. His main example is the square of the binomial, $(a - b)^2$, where the product of $-b$ by itself is taken to be plus b^2. He criticizes the usual arguments provided to justify that *minus* times *minus* is *plus* for not being general, nor cogent enough; the argument based on grammar, for instance, does not apply to the French language, and other arguments are mere analogies devoid of mathematical content:

"Il est à remarquer que la plûpart des operations de l'Algebre sont fondées sur des propositions de Geometrie et d'Arithmetique; et que par consequent on ne peut pas demontrer par ces operations, les mesmes propositions qui leur ont servy de preuve; ... En voicy un exemple. On trouve par le calcul de l'Algebre, que $[(A - B)^2 = A^2 + B^2 - 2AB]$ ... , et l'on prend dans ce calcul B^2, pour le produit de $-B$ par $-B$, ... ce qui est fort surprenant; ... Quelques-uns dissent que cela procede, de ce que deux negations valent une affirmation; mais c'est une Regle de Grammaire, qui est mesme fausse dans la Grammaire Françoise; et dans ce calcul, on ne nie point, mais on multiplie. D'autres disent, que moins moins vaut autant que plus plus; ce qui est inconcevable, bien loin d'estre clair et évident. Il est donc necessaire de prouver la bonté de cette operation, puis qu'elle ne s'établi pas d'elle-mesme. La preuve s'en fait par la septiéme du second des Elemens d'Euclide". (Mariotte, 1992 : 82-3).

He concludes that the only proof deserving such a name is by Proposition 7, Book II of Euclid's *Elements* (Mariotte (1992), p. 82-3). This proposition demonstrates that the square of any line plus the square of some part of it equals the square of the remaining part plus twice the rectangle whose sides are the whole and the part (see Figure 1). This proposition is the only way to explain why – B square equals B square: "c'est la raison pour laquelle il faut prendre B^2 pour le produit de $-B$ par $-B$" (Mariotte (1992), p. 83).

To our eyes and minds, already well used to the existence of formal lan-

Elements II, 7th:

$A^2+B^2 = (A-B)^2+2AB$

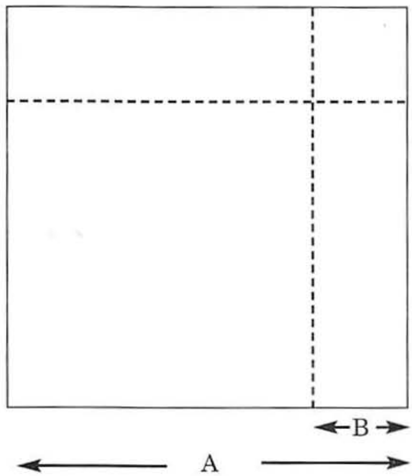


Figure 1. Euclid's *Elements*, Book II, Proposition 7

Proposition II 7 of the *Elements* proves that $A^2 + B^2 = (A - B)^2 + 2AB$ by means of the diagram above.

guages made up of abstract symbols, and to the power of the algebraic language, the mere creation and use of algebraic symbols may look equivalent to the creation of a new mathematical language, new mathematical objects and new standards of proof. Yet, as this trivial example demonstrates it is one thing to introduce algebraic symbols (even if accompanied with precise rules for handling them), and a very different thing to turn them into a self-sustaining language —that is to say, a language that is able to support by itself (without having recourse to geometric objects and results) the weight of mathematical demonstrations. In other words, this example suggests that the so-called algebraic "style" of the 17th century coexisted with a philosophy of mathematics in which geometry kept a *fundamental* role— fundamental in a literal as well as in a metaphorical sense. We will examine now this intermediate stage in the development of the notion of algebraic "language" by looking into John Wallis's understanding of its nature.

Wallis on algebra

Some of the prominent features in Condillac's praise for algebra are already in John Wallis. Wallis stressed that discovery and proof go along separate ways. Although he did not explicitly identify discovery with analysis, his defense of the greater relevance of discovery over synthetic presentation and demonstrations (on which more below) is fully consistent with Condillac's. According to Wallis, algebra was (among other things) the art of resolving or "dissolving" what is supposed to be compounded or made up (Wallis, 1685 : 1). This included prominently a theory of equations whose main purpose was the "computation and management of proportion" in an abstract, general way.[6] Between the two authors, however, we find substantial and highly revealing differences that concern as much their philosophy of algebra as their understanding of the historical development of the subject, and even Wallis's views on some ethical implications derived from specific mathematical styles.

For one thing, Wallis's praise of algebra as a tool of discovery was ambiguous in its historical significance. Condillac explicitly criticized classical thinkers and mathematicians for having a poorer language and a less sophisticated methodology than the ones he was promoting. He was a man of the Enlightenment, looking forward to an unending material, moral, and scientific progress, and looking down onto an unsophisticated past unaware of the right method in philosophy. To Wallis, on the other hand, the past was still important to legitimize algebra and algebraic demonstrations. Following Viète, Oughtred, and most early users of algebraic symbols, Wallis presented algebra as an "art of invention" that had in all probability an ancient pedigree. In many places of his *Treatise of Algebra*, Wallis sets forth his view that the mathematicians of the past did know of and used some secret method of discovery:

"It is to me a thing unquestionable, That the Ancients had somewhat of like nature with our Algebra, from whence many of their prolix and intricate Demonstrations were derived. ... But this their Art of Invention, they seem very studiously to have concealed". (Wallis, 1685 : [2], 3).

He also drew a sort of chronology of the transmission of such secret art: from the most ancient civilizations of India algebra came to the Perses, from them to the Arabs, who used it before the ancient Greeks. From the Arabs, by means of the "Saracens and Moors" it went to Spain, and hence to England (how or who moved it from the Iberian peninsula to England, he does not say) (Wallis (1685), p. [2], 4). So Wallis promotes algebra because of its heuristic power, and because he deems discovery to be more important than proof (more about this below), but he promotes it under the example, so to speak, of the great mathematicians of the past.

[6] Wallis highly praised Thomas Harriot for showing how equations of higher degrees were composed or made up of first-degree equations and other "more simple equations" (Wallis somehow misrepresented and magnified Harriot's contribution, but this is certainly irrelevant here). This was "the great key that opens the most abstruse mysteries in algebra". See Wallis (1685), p. 198-9.

Wallis's praise for algebra gets undeniable moral overtones. As his references to the "secrets concealed" by the mathematicians of the past suggest, Wallis takes for granted that mathematicians not only may wish to conceal their working methods but actually do conceal them. There is an explicit moral reproach in Wallis's attack on the classical, elaborate, geometrical demonstrations by reduction to absurdity, and in his praise for the algebraic style: this is a "humbler" way of practicing mathematics because it provides ways and means that may help readers in making mathematical discoveries. Wallis criticizes classical mathematicians who made public only prolix and intricate geometrical demonstrations that rest on "the Pompous ostentation of Lines and Figures" (Wallis, 1685 : 3, 298). In a highly interesting passage, he uses a mechanical metaphor. Quoting Pedro Nunes, who had quoted Aristotle, Wallis identified such classical mathematicians to the technicians of old times who contrived marvelous machines "but [did] conceal the Artifice [making the machines run], to make them the more admired" (Wallis, 1685 : 3). Wallis used metaphorically mechanical machines to suggest that there may be a hidden trick in demonstrations that look difficult, and to illustrate the message that the reader should not unduly admire discoveries that may seem out of the powers of ordinary persons. Condillac used the same image of a mechanical machine but to make quite a general epistemological point, not a moral one. Rather than a predecessor of Condillac, Wallis seems to be here a successor of Francis Bacon, who had opposed terse and honest scientific prose to the ornate, pompous, and contrived style that hides rather than conveys the author's meaning. As is well known, this highly popular idea in the early years of the Royal Society found expression in such works as John Wilkins's *Essay towards a Real Character and Philosophical Language* (1668) and Thomas Sprat's *History of the Royal Society* (1667).[7]

According to Wallis, the most relevant feature of the algebraic style (one on which he insisted in different places of his *Treatise*) was the "short and convenient way of Notation; whereby the whole process of many Operations is at once exposed to the Eye in a short Synopsis."[8] That is to say, Wallis's algebraic "language" was a shorthand-like set of signs rather than a true formal language. He praised and adopted Oughtred's numerous notational conventions,

$$\begin{array}{ll} \mathrm{A}+\mathrm{E}=\mathrm{Z} & \qquad \mathrm{A}^2+\mathrm{E}^2=Z \\ \mathrm{A}-\mathrm{E}=\mathrm{X} & \qquad \mathrm{A}^2-\mathrm{E}^2=X \\ \mathrm{A}\cdot\mathrm{E}=\text{Æ} & \qquad \mathrm{A}^2\cdot\mathrm{E}^2=\text{Æq}\text{ , etc,} \end{array}$$

[7] Jones (1961), p. 48-9. The classical locus for Francis Bacon's views on language is the *Novum organum*, aphorisms 43, 59, and 60. H. Pycior makes reference to Bacon in relation to the "symbolical" algebra of W. Oughtred and Wallis, although she does not points to the moral content in Wallis's position; see Pycior (1997), p. 46, 121.

[8] Wallis (1685), p. [4]. As Pycior has shown, this idea was already prominent in Oughtred; see Pycior (1997), p. 40ff.

by stressing that they "do save a great Circumlocution of words (each letter serving instead of a Definition;)". With such conventions, says Wallis, Oughtred manages to have "a great deal of very good Geometry brought into a very narrow room" (Wallis, 1685 : 67). Abbreviation of long expressions and use of "marks", "notes" or "signs" that speak "to the eye" and aid the memory are the main virtues of Wallis's algebraic notations:

"[such notations] I often find very useful ... for assisting the Fancy, and easing the Memory, and bringing the whole Process to as narrow a prospect as may be; which thereby becomes intelligible, with more much ease than when involved in a multitude of Words, and long Periphrases". (Wallis, 1685 : 68).

Wallis is well aware that algebraic notations have also the advantage of being "general" or "abstract", which means that they apply to lines, figures and to any other quantities (Wallis, 1685 : 69, 298-9). Yet, when he opposes algebraic notations to geometric demonstrations he never claims that they constitute a superior language. In keeping with 17th-century theories of language, he consistently emphasizes that algebraic notations make easier the unadulterated access of mathematical ideas and judgments to the mind by eliminating words and streamlining language generally. Rather than new kinds of words, algebraic notations are considered graphic tricks that eliminate words and "long periphrases" because they convey ideas through the "eye" directly to the mind. Wallis stresses the shorthand quality of his notations by demanding that they remind the reader of the quantities they represent (Wallis, 1685 : 68).

In 1654 Wallis's colleague Seth Ward (1617-1689), then the Savilian Professor of Astronomy in Oxford, elaborated the standard view on the algebraic style in the context of the so-called Webster-Ward debate.[9] Wallis and Ward, along with Boyle, Hooke, Thomas Willis, John Wilkins, and others, belonged to the circle of natural philosophers that met at Wadham College, Oxford, in the early 1650s. John Webster (1610-1682) was a Puritan minister who had studied chemistry and medicine and served as surgeon as well as chaplain with Cromwell's army. Promoting the radical reform of the universities, in his *Academiarum Examen* (1654) he defended the generalized introduction of new subjects, such as the study of hieroglyphics, alchemy, and algebra, among other things. Besides the interest he had in the foregoing subjects *per se*, Webster claimed that all of them involved symbolic languages of a new kind that could be improved by the help of grammar. Grammar, also in need of reform, was defective in not paying to such languages the attention they

[9] On the debate, see Debus (1970), which includes facsimile reprints of Webster's *Academiarum Examen* and Ward's *Vindiciae Academiarum*.

deserved. They might in turn be useful to make progress towards a universal language (Debus, 1970 : 106-8). As a chemical philosopher of Hermetic leanings, Webster believed in an Adamic, primordial, natural language that had been lost in the fall but could in some measure be restored by a deep and pious reform of learning (*ibid* : 110-4). His pamphlet found answer in Seth Ward's *Vindiciae Academiarum* (1654), prefaced by John Wilkins, in which Wallis's views on algebra are manifest. Ward, assuming a typical 17th-century philosophy of language, is adamant in opposing language and grammar to anything involving symbols—and algebraic notations in particular. In fact, symbols and graphic signs he takes to be as incompatible with language as life with death:

"to say that by introducing [signs], either Grammar or Languages should be advanced, it were as mysticall as to affirme, that the day light is advanced by the coming of the night, or that he would kill a man for his preservation". (Debus, 1970 : 212).

For one thing, Ward sees the handling of hieroglyphics and cryptographic devices opposed to language because the aim and main use of such signs is the concealment of information, while language aims to illustrate, explicate, and convey information. As for algebra, he uses almost verbatim words that Wallis will use in his *Treatise*. Since the main purpose of algebraists is to eliminate words, Ward says, it is absurd to put them now under the authority of grammarians. Algebraic notations were meant to eliminate words because they enable us "to behold, as it were, with our eyes ... [a] long series of ... intricate Ratiocination". They speak, through the eye, directly to the mind, like pictures. Therefore, as Ward put it, "[the invention of algebraic symbols] was a designe perfectly intended against Language and its servant Grammar." (Debus, 1970 : 213) The argument holds for algebra as well as for any other use of symbols in philosophy (Ward mentions here the Cabbalists and other schools) or mathematics: if they are useful in improving knowledge it is because they provide an alternative to words, "a shorter, and clearer cutt to the understanding" (Debus, 1970 : 214). Ward also discusses the notion of a "universal language". Interestingly he acknowledges that his getting acquainted with algebraic notations was what prompted his interest in the matter. He envisaged the application of symbols to the designation of all things. First he envisaged it in a direct and simple way which required an "almost infinite" number of different symbols, one for each existing thing or notion, and then in a more sophisticated way by resolving words in the most "simple notions" and using different symbols only for the "simple notions" (Debus, 1970 : 214-6). In any case, Ward's use of symbols was always imagined not as providing a new syntax, but as an improvement in the communication of pre-existing notions and syllogisms. Ward's views were in all probability discussed with Wallis in the weekly meetings they held at Wadham College. They show how much the mathematician's understanding of the role of algebraic notations was shaped by contemporary views on the nature of language.[10]

Proving by algebraic notations?

As we shall see now, Wallis's views on the implications of algebra for the philosophy of mathematics are hardly consistent, and even his practice is difficult to conceptualize. On the one hand, Wallis said that the introduction of algebra did *not* fundamentally change mathematical objects and demonstrations. As he put it, algebra furnishes convenient notations "without altering the manner of demonstration, as to the substance" (Wallis, 1685 : [4]). Since algebraic notation, according to Wallis, was something like a tachigraphy, or a series of tricks to abbreviate expression (rather than a language proper), it was not too inconsistent for him to claim that the new symbolic style provided demonstrations as good as the classical geometrical ones, since they were but abbreviations of, say, Archimedes's. Thus, after proving by incomplete induction the formula for the sum of the squares of an arithmetical progression [$\Sigma_{1\leq k\leq n} k^2 = n(n + 1)(2n + 1)/6$], Wallis insisted in stressing that it was equivalent to Proposition 10 of Archimedes's *On spirals*: "So that [Archimedes's] Proposition is, for substance, the same with mine, though otherwise expressed" (Wallis, 1685 : 300).[11] Again, a few pages later: "[I] represented [classical demonstrations] in a manner more obvious to be apprehended." (Wallis (1685), p. 304) In addition, by way of rationale for promoting the new style, he argued that "because as [the demonstrations] lye in Archimedes, they seem very perplexed; I thus digested them into a brief Synopsis, (that they might the better be apprehended)" (Wallis, 1685 : 301). Wallis set forth this idea in other places of the *Treatise* (for instance when introducing Oughtred's symbols, on page 68), but stressed this idea at its very beginning, in the Preface, when he introduced "specious arithmetick" as the art that

"doth (*without altering the manner of demonstration, as to the substance*) furnish us with a short and convenient way of Notation; whereby the whole process of many Operations is at once exposed to the Eye in a short Synopsis".Wallis (1685), p. [iv]. [12]

[10] Pycior also mentions the Webster-Ward debate when dealing with Wallis's understanding of algebra. Not discriminating between Wallis's notations and a full-fledged language, she only found in the debate evidence of the "sharp dividing line between the old prose and the scientific language advocated by Bacon and his English followers" (Pycior (1997), p. 115).

[11] The idea was reiterated on pages 299 and 301.

[12] (emphasis added).

Yet, on the other hand, Wallis was not always consistent with such a view, for he used demonstrations that were not classical, nor could be cast in classical terms. He was most daring in using incomplete induction and in taking advantage of the formal features of algebraic notations. In any case, even in such demonstrations we find evidence that Wallis perhaps did *not* realize how powerful algebra was bound to be, if understood as a self-sustaining language. We shall deal mostly with his understanding of algebraic demonstrations, where considerations of formal and graphical symmetry play a crucial role, but his musings on the introduction of new numbers and new curves by means of algebra deserve a brief mention.

Historians of mathematics often praise Wallis for his explicit defense of negative and imaginary numbers, which is taken as evidence that he was promoting algebra against geometry. As we have shown elsewhere, however, Wallis's discussed such novel quantities to show that the handling of such imaginary quantities is not "either Unuseful or Absurd; when rightly understood." (Wallis, 1685 : 265). [13] Wallis's arguments do not involve a defense of such new numbers *per se*. He does not vindicate the freedom of the mathematical mind to create new objects (numbers in this case) out of the formal power of the language of algebra. Rather to the contrary, he acknowledges explicitly that strictly speaking such quantities make no sense in themselves. When "defending" them, Wallis only points to specific problems in which some geometrical and physical interpretations of such quantities are highly useful. One of his examples is that if we win to the sea a square surface of 1600 square perches, its side is 40 perches, but if we lose it to sea, we have now –1600 square perches, whose side may be represented by $\sqrt{-1600}$ (Wallis, 1685: 265). There are a few other examples, but the point is always that by adding explanations to familiar and "real" situations, and by "reading" geometrical problems from some unusual perspective, "impossible" quantities may have some meaning and may help the mathematician by providing a measure of the "impossibility" involved in the problem (Wallis, 1685 : 264-72).

Wallis's considerations on the nature of curves defined algebraically are even more interesting. In the 1650s, after the publication of Wallis's *Arithmetica infinitorum* (1656), William Neil published the first rectification of a curve. Following a hint in Wallis's book, Neil showed that finding the arc of semi-cubic parabolas (which we write today $ky^2 = x^3$) could be reduced to finding the quadrature of (simple) parabolas (Whiteside, 1960 : 328-30). Wallis, who was always ready to magnify the achievements of English mathematicians, is surprisingly tepid in his appraising of Neil's rectification. The reason is, as he explains in a let-

[13] See A. Malet, "Notions of number, 1585-1685", forthcoming.

ter to the secretary of the Royal Society, Henry Oldenburg, that semi-cubic parabolas are not curves to be taken seriously. They are merely defined by an algebraic equation. Surely, we can draw them by means of the equations, but have no other recognizable properties. Such curves, which are devised purposefully taking into account the properties of their defining equation, are curves that mathematicians should not care too much about, Wallis says. They are not interesting in themselves and deserve not to be taken seriously because they lack their own geometric properties. Wallis mentioned the cycloid as an example of "recognized curves" whose study (i.e., whose rectification, quadrature, and so on) deserved the highest praise and the full attention of the best mathematicians.[14] Wallis's views are tantamount to saying that the algebraic definition of curves has no mathematical interest—or to be more precise, that such curves whose definition is but an algebraic equation have no mathematical significance. We have evidence that Wallis's views on curves were no idiosyncratic. We find them almost verbatim in a letter of Christiaan Huygens to Leibniz as late as in 1691. Huygens criticizes the "contriving" (*forger*) of curves, meaning the invention of curves whose algebraic equations ensure them certain properties. And he criticizes also the study of such curves, which have no other utility than showing how to solve some exercises. Such curves not worth the serious mathematician's attention, Huygens opposes to the curves that are truly interesting:

"Je ne voudrois jamais m'amuser à ces differentes natures de chaines, que Mr. Jo. Bernouilly propose ... Il y a de certaines lignes courbes que la nature presente souvent à notre vue, et qu'elle decrit pour ainsi dire elle mesme, lesquelles j'estime digne de consideration, et qui d'ordinaire renferment plusieurs proprietez remarquables, com l'on voit au Cercle, aux Sections coniques, à la Cycloide, ..."

In the preparatory draft for the letter, Huygens adds: "De telles lignes meritent ... qu'on se les propose pour exercice, mais non pas celles qu'on forge de nouveau seulement pour y employer le calcul geometrique." In the final version the message is the same, complemented with a reference (also negative) to algebraic problems in number theory.[15] As we shall see presently, Wallis also dismissed the algebraic problems in higher arithmetic as a frivolous divertimento.

Wallis's reflections on algebraic demonstrations were mostly prompted by Fermat's harsh, pointed criticism to his demonstrations in his *Arithmetica infinitorum*, some of which include incomplete induction.

[14] Wallis to Oldenburg, 4 October 1673, in Hall and Hall (1965-1986), X, p. 280.

[15] The letter reads: "Mais d'en forger de nouvelles, seulement pour y exercer sa geometrie, sans y prevoir d'autre utilité, il me semble que c'est *difficiles agitare nugas*, et j'ay la mesme opinion de tous les problemes touchant les nombres." See Huygens to Leibniz, 1 September 1691, in Huygens (1888-1950), X, p. 128, 132-3.

That is to say, they involve the parameter n, integer > 0, and Wallis "proves" them by checking the truth of the result for small values of n and by analyzing the reasons why the result is true. Critiques to Wallis's methods and answers to them traveled in letters that Wallis gathered and published in book form shortly thereafter as *Commercium epistolicum ... Inter Nobilissimos Viros [Lord Brouncker, Fermat, Frenicle, Wallis, van Schooten, et al.]* (1658). Fermat and some of his friends (Frenicle acting most of the time as Fermat's mouthpiece) were adamant in denying the status of demonstrations to many arguments Wallis used there to support his main results. Their objections pointed to Wallis's incomplete induction, but also extended to the use of "analytical formulae", which they opposed to the elegance of Euclid, Apollonius, and classical mathematics generally.[16] Anticipating the idea that was to reappear in his *Treatise of Algebra*, Wallis's answer stressed that his demonstrations were most appropriated to illuminate the way in which he gained the results. Rather than to admire Archimedes for his elegant demonstrations, says Wallis, we should blame him for hiding his method of investigation.[17]

The other dominant theme in the Fermat-Wallis exchange was their sharply contrasting views about (what we nowadays call) number theory. Fermat's eager and repeated invitations for Wallis to join him in his arithmetical researches were not only declined but the researches themselves dismissed as useless and unimportant (Wallis, 1658 : 761, 782). Fermat and Frenicle rejoined that results in higher mathematics have no mechanical uses, and that only the lower, easier, and less prestigious parts of mathematics were used in surveying, gauging, and similar jobs. They forcefully set forth the notion that mathematical truths must be pursued for their subtlety and perfection, and because it is most fitting for the human mind to look after the pure truth, no matter in what subject it happens to be embodied—to no avail (Wallis, 1658 : 810-1, 858, 835, 840). Wallis's (and van Schooten's, Hudde's and Huygens's among others) unwillingness to get into number theory, therefore, was a consequence of an implicit philosophy of mathematics shared with many other influential mathematicians of his time. In this philosophy, the algebraic style was important if it could be used to solve certain families of problems of a geometrical sort. When the development of algebra depended on attacking problems that Wallis and company deemed not to be important enough, algebra was left aside.

It is time now to discuss the inherent limitations algebraic demonstra-

[16] For details on Wallis's incomplete induction, see Scott (1938), p. 35-60. The debate is found in Wallis (1658) (also printed in Fermat (1894-1912)). For Fermat's criticism, see Wallis (1658), p. 761, 858; see also Fermat's "Remarques sur l'arithmétique des infinis du S. J. Wallis", in Fermat (1894-1912), II, p. 347-353.
[17] Wallis (1658), p. 777-789, particularly 781-782.

tions had according to Wallis. Many years after the original exchange of views prompted by the *Arithmetica infinitorum*, and when Fermat had been dead for many years as well, Wallis came back to Fermat's criticism in his *Treatise of Algebra* —proof of how important the issue was to him. The least that can be said about Wallis's views on algebraic demonstrations as set forth in the *Treatise* is that they are hardly consistent. As we saw, repeating his former answer to Fermat, Wallis claimed that algebra (and its adjunct, incomplete induction) was a tool for making discoveries, as opposed to producing proofs, and that it should be appreciated according to it. Yet, he also claimed that algebra was not essentially or fundamentally modifying the structure and arguments of classical mathematics. As we shall see, what lies behind these conflicting claims is that Wallis apparently was not in full possession of the notion of what we call now a formal demonstration.

Wallis illustrated his views with examples that provide clear evidence of what I call the formal weakness of Wallis's algebraic notations. In page 307 of his *Treatise* he printed the arrays:

0			0				0				0					0					
	1			1				1				1					1				
1		0	1		2		1		6		1		14			1		30			
	1			3		0		7		6		15		36			31		150		
2		0	4		2		8		12		16		50		24	32		180		240	
	1			5		0		19		6		65		60			211		390		120
3		0	9		2		27		18		81		110		24	243		570		360	
	1			7		0		37		6		175		84			781		750		120
4		0	16		2		64		24		256		194		24	1024		1320		480	
	1			9		0		61		6		369		108			2101		1230		120
5		0	29		2		125		30		625		302		24	3125		2550		600	
	1			11		0		91		6		671		132			4651		1830		
6		0	36		2		216		36		1296		434			7776		4380			
	1			13				127				1105					9031				
7			49				343				2401					16807					

They feature the natural numbers (1, 2, 3,) and their differences; their squares (1, 4, 9,) and their 1st- and 2d-order differences; their cubes and all their differences, and so on and so forth. Wallis remarks that the last equal differences follow an obvious law. Then he poses himself the rhetorical question, "Is it possible to prove such law generally?" The remarkable answer is highly instructive. According to Wallis, the "observation" (*sic*) of the arrays themselves is enough to convince anybody of such result:

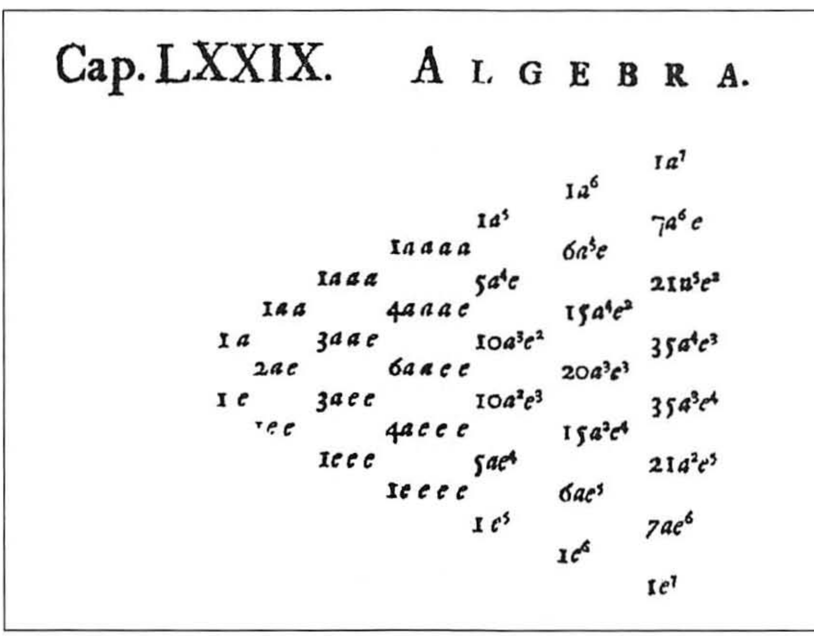
"Such observation would be looked upon, as sufficiently instructive; since there is no reason of Suspicion, why it should not so continually proceed: But [there is] reason rather to believe, that there is, in the nature of Number, a sufficient ground of such sequel".

Wallis goes on then to explain that a "large treatise" could be filled with a "solemn Process of Demonstration" proving the result for the arithmetical progression of natural numbers (1, 2, 3, ...), and then for the squares of them, and so on:

"And at length conclude (for to that he must come at least, unless he would be infinite,) That we have reason to judge in like manner, of consequent Powers, (which concludes the induction)". (Wallis, 1685 : 307).

We can do nothing, according to Wallis, to prove in general the rule that we "observe" in the foregoing arrays. As Wallis explains, all the best and most demanding mathematician can do here is a complete demonstration for *every* power, k, in the series $\{n^k\}_n$. In other words, Wallis seems to have no conception of what would be an algebraic proof for this general series that has k as a parameter. Yet, Wallis has no doubts about the truth and generality of the result and claims them to be obviously derived from the study of the foregoing arrays. The source for his confidence is to be found in the graphic disposition of the symbolic arrays. Wallis perceives in them all the hints he needs to generalize his observations, while he cannot imagine the formal mechanism that provides what we call nowadays an argument by complete induction.

In providing more examples to fully explain his position, Wallis deals with the powers of a binomial by means of the following array:


Cap. LXXIX. ALGEBRA.

$1a^7$
$1a^6$
$1a^5$ $7a^6e$
$1aaaa$ $6a^5e$
$1aaa$ $5a^4e$ $21a^5e^2$
$1aa$ $4aaae$ $15a^4e^2$
$1a$ $3aae$ $10a^3e^2$ $35a^4e^3$
$2ae$ $6aaee$ $20a^3e^3$
$1e$ $3aee$ $10a^2e^3$ $35a^3e^4$
$1ee$ $4aeee$ $15a^2e^4$
$1eee$ $5ae^4$ $21a^2e^5$
$1eeee$ $6ae^5$
$1e^5$ $7ae^6$
$1e^6$
$1e^7$

First, he states the formation laws for this table: (1) any coefficient in the table is the sum of the two coefficients anteceding it; (2) the powers of a and e appearing in any term are the ones appearing in the 2 terms of the foregoing column anteceding it. His comments on what this array means for the powers $a + e$ are worthwhile quoting in full. He allows that any specific power can be demonstrated algebraically (by "specious multiplication"), and stresses that the production of the first powers and the observation of the array they produced is usually assumed to be a conclusive demonstration:

"Now each of these steps, [meaning $(a + e)^2$, $(a + e)^3$, and so on] may be singly demonstrated by a Specious Multiplication of $a + e$ into itself, which will produce the Square ... ; and then of this into $a + e$, which will produce the Cube ...: and so onward, (by continual Induction.) But most Mathematicians that I have seen, after such Induction continued for some few Steps, ... are satisfied (from such evidence,) to conclude universally, *and so in like manner for the consequent Powers*. And such Induction hath been hitherto thought ... a conclusive Argument". (p. 308). (Emphasis in the original).

What would the answer be, for anyone not satisfied with such an argument? Interestingly, Wallis also says that he cannot think of a general proof for the development of $(a + e)^n$. The only alternative Wallis can think of is to "continue the Process (by continual Multiplication into $a + e$,) as far as they please; and then content themselves ... with a particular conclusion (for they prove no more,) that *it holds true as to so many steps*; and rest there."[19] Therefore, Wallis sees no real alternative to his "inductions" based on the regularity of algebraic notations. To quote him once more, any "inductive" demonstration of his

"I always pursue so far (by regular demonstration...) till it lead me into a regular ... Process; ... And without this, we must be content to rest at particulars (in all such kind of Process,) without proceeding to the Generalls". (Emphasis in the original).

By way of conclusion let me stress that the weakness of Wallis's algebra *qua* formal language is fully consistent with his emphasis on algebra as a tool of discovery. But it is also consistent with his view that algebra does not produce "new" demonstrations, or that algebra does not alter the substance of the classical demonstrations. Finally, and perhaps more significantly, such a weakness is also consistent with the notion that algebra was a tachigraphy that "condensed" mathematical arguments and "offered them at one glance". In Wallis, algebraic notations work through the visual display of symmetries and regularities. It was such graphic qualities of his notations that played a crucial role in the new algebraic style of the 17th century. Not yet a fully self-sustaining formal language, algebraic notations were more like an alphabet than a language, and as such, considerations of symmetry and design could not be avoided.

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